§8 Artin Rings

Artin ring = ring satisfying d.c.c. on ideals

(因为简单！)

Prop 8.1. In an Artin ring every prime ideal is maximal.

Pf: $\forall\ \mathfrak{p} \triangleleft A$: prime $\Rightarrow$ $B := A/\mathfrak{p}$ = Artin integral.

$\forall\ x \in B \setminus 0$. consider

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

$\exists\ n$ s.t. $\quad (x^n) = (x^{n+1})$

$\Rightarrow\ x^n = x^{n+1} y$ for some $y \in B$

$\Rightarrow\ xy = 1 \ \Rightarrow\ B = \text{field} \Rightarrow \mathfrak{p} = \text{maximal}$ $\square$

Cor 8.2. $A = \text{Artin} \Rightarrow \mathrm{Nil}(A) = \mathrm{Rad}(A)$.

Prop 8.3. An Artin ring has only finite number of maximal ideals.

Pf: $\forall\ \mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_{r+1}$ different maximal ideals.

$$\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r \neq \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{r+1}.$$

(Suppose not. Then $\mathfrak{m}_{r+1} \supseteq \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$.

$\Rightarrow \mathfrak{m}_{r+1} \supseteq \mathfrak{m}_i$ for some $i=1,\cdots,r$ $\lightning$)

Suppose it has ∞'ly many maximal ideals

$$\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \cdots\cdots$$

$$\Rightarrow \mathfrak{m}_1 \supsetneq \mathfrak{m}_1\mathfrak{m}_2 \supsetneq \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supsetneq \cdots\cdots \quad \lightning$$

Prop 8.4. A = Artin ring $\Rightarrow \exists\ n$ s.t.

$$\mathrm{Nil}(A)^n = 0.$$

Pf: $\mathrm{Nil}(A) \supseteq \mathrm{Nil}(A)^2 \supseteq \cdots\cdots$

$$\Rightarrow \mathfrak{a} := \mathrm{Nil}(A)^n = \mathrm{Nil}(A)^{n+1} = \cdots$$

Suppose $\mathfrak{a} \neq 0$.

$$\Sigma := \{\, \mathfrak{b} \triangleleft A \mid \mathfrak{a}\mathfrak{b} \neq 0 \,\} \neq \phi$$

Artin $\Rightarrow \exists\, \mathfrak{I}$ = minimal in Σ

$\mathfrak{a}\mathfrak{I} \neq 0 \Rightarrow \exists\, x \in \mathfrak{I}$ s.t. $x\mathfrak{a} \neq 0$.

$\Rightarrow \mathfrak{I} = (x)$

$x\mathfrak{a}^2 = x\mathfrak{a} \neq 0$ & $x\mathfrak{a} \subseteq (x)$

$\Rightarrow x\mathfrak{a} = (x)$

$\Rightarrow \exists\, y \in \mathfrak{a}$ s.t. $xy = x$

$\Rightarrow x = xy = xy \cdot y = \cdots = xy^m = 0$ ↯.

$$\dim(A) := \sup\{ n \mid \underbrace{\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ prime ideals of } A}_{\text{chain of prime ideal}} \}$$

length

Thm 8.5 $\quad A$ = Artin $\Leftrightarrow$ A = Noetherian & $\dim A = 0$.

Pf $\Rightarrow$): (8.1) $\Rightarrow \dim A = 0$

(8.3) Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ be the distinct max. ideals

$$\overset{(8.4)}{\Rightarrow} \quad m_1^k \cdots m_n^k \subseteq \left(\cap m_i\right)^k = Nil(A)^k = 0, \quad k \gg 0.$$

$$\overset{(6.11)}{\Longrightarrow} \quad A = \text{noetherian}.$$

$\Leftarrow$): $\quad 0 = \bigcap_{i=1}^{n} q_i$ primary decomp.

$m_i := \sqrt{q_i}$ maximal ($\dim A = 0$)

$\Rightarrow \exists k$ s.t. $m_i^k \subseteq q_i$

$$\Rightarrow \prod_i m_i^k \subseteq \bigcap_i m_i^k \subseteq \bigcap_i q_i = 0$$

$$\overset{(6.11)}{\Longrightarrow} \quad A = \text{Artin}.$$

Fact: Let (A, m) be an Artin local ring.

- m is the only prime ideal
- $m = Nil(A) = Rad(A)$
- $m^n = 0$ for $n \gg 0$
- $A = A^\times \amalg m$

e.g. $\mathbb{Z}/p^n\mathbb{Z}$

Prop 8.6. (A, m) = Noetherian local ring. Then

i) $m^n \neq m^{n+1}$ $\forall n$

or

ii) $m^n = 0$ for some m.

In this case. (A, m) = Artin local ring.

Pf: Suppose $m^n = m^{n+1}$ for some m.

Nakayama $\Rightarrow$ $m^n = 0$

$\Rightarrow$ $m \subseteq \mathfrak{p}$ $\forall$ prime $\mathfrak{p}$

$\Rightarrow$ A = artin. $\square$

Thm 8.7 (Structure thm for Artin rings)

An Artin ring A is uniquely (up to iso) a finite direct product of Artin local rings.

Pf: $\{m_1, \cdots, m_n\} := \operatorname{Spec} A$.

Pf of (8.5) $\Rightarrow$ $\prod_{i=1}^{n} m_i^k = 0$ $k \gg 0$.

$$(1.16) \Rightarrow m_i^k + m_j^k = A$$

$$\Rightarrow \bigcap_{i=1}^{n} m_i^k = \prod_{i=1}^{n} m_i^k = 0$$

$$\Rightarrow A \xrightarrow{\sim} \prod_{i=1}^{m} (A/m_i^k)$$

Example: $A = k[x_1, x_2, x_3, \cdots] / (x_1, x_2^2, x_3^3, \cdots)$

$\# \operatorname{Spec} A = 1$ & $A \neq$ Noether & $\neq$ Artin.

Prop 8.8. (A, m) = Artin & local. TFAE

i) $\forall \mathfrak{a} \triangleleft A$ is principle

ii) m is principle

iii) $\dim_k (m/m^2) \leq 1$.

Pf: i) $\Rightarrow$ ii) $\Rightarrow$ iii) clear.

iii) $\Rightarrow$ i). 1° $\dim_k (m/m^2) = 0 \Rightarrow m = m^2 \Rightarrow m = 0$

$\Rightarrow A = \text{field} \Rightarrow \checkmark$

2° $\dim_k m/m^2 = 1 \Rightarrow m = (x) + m^2 \overset{(2.8)}{\Rightarrow} m = (x)$

$\forall\, \mathfrak{a} \triangleleft A$ (assume $\mathfrak{a} \neq (0)$ & $\neq (1)$.)

$\Rightarrow \mathfrak{a} \subseteq \mathfrak{m}^r$ & $\mathfrak{a} \nsubseteq \mathfrak{m}^{r+1}$

$\Rightarrow \exists\, y = a x^r \in \mathfrak{a} \setminus \mathfrak{m}^{r+1}$

$\Rightarrow a \notin (x) \Rightarrow a \in A^\times$

$\Rightarrow x^r = a^{-1} y \in \mathfrak{a} \Rightarrow \mathfrak{m}^r \subseteq \mathfrak{a}$

$\Rightarrow \mathfrak{a} = \mathfrak{m}^r$ $\square$

Example: $\mathbb{Z}/p^n\mathbb{Z}$ & $k[x]/f^n$ f = irr.

$k[x^2, x^3]/(x^4) \cong k[y,z]/(y^2, yz, z^2)$

$= k \oplus \underbrace{ky \oplus kz}_{\mathfrak{m}}$